Look at :

$$\begin{cases} \tilde{x}(t) = \tilde{x}(0)\cos\omega t + \dfrac{\tilde{p}(0)}{m\omega}\sin\omega t \\ \tilde{p}(t) = -m\omega\,\tilde{x}(0)\sin\omega t + \tilde{p}(0)\cos\omega t \end{cases}.$$

$$\Leftarrow \quad m\frac{d\tilde{x}}{dt} = \tilde{p}, \quad \text{just like C.M.}$$

But, $\langle\tilde{x}\rangle$ and $\langle\tilde{p}\rangle$ are "not" oscillating. !!!

although $\tilde{x}(t)$ and $\tilde{p}(t)$ look like oscillating.

|| NOTE : $\langle\tilde{x}\rangle = 0$, $\langle\tilde{p}\rangle = 0$. for all $|n\rangle$.

Q. Can we find a "Quantum" state

that behaves just like classical $\langle x\rangle$ and $\langle p\rangle$?

* Coherent States.

This is the one.

why do we need this ?

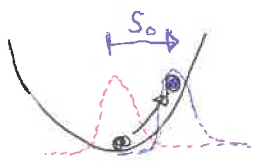
- We live in a "classical" world,

But we want to control a "Quantum" world.

: We need a "bridge" !

the easiest way to make the coherent state

$\rightarrow$



$$|s_0\rangle = J(s_0)\,|0\rangle$$

move the ground state to s_0.

wave function $\psi_{s_0}(x) = \psi_0(x - s_0)$. $\quad \| \quad \langle x|J(s_0) = \langle x - s_0|$

observables :

$$\langle s_0|\tilde{x}|s_0\rangle = \langle 0|J^\dagger(s_0)\,\tilde{x}\,J(s_0)|0\rangle = s_0$$

$$\langle s_0|\tilde{p}|s_0\rangle = 0.$$

$$\langle s_0|H|s_0\rangle = \langle 0|\frac{\tilde{p}^2}{2m}|0\rangle + \frac{1}{2}m\omega^2\langle 0|(\tilde{x}+s_0)^2|0\rangle.$$

(cont.)

$$\langle s_0|H|s_0\rangle = \langle 0|H|0\rangle + \frac{1}{2}m\omega^2 s_0^2$$

$$= \underbrace{\frac{1}{2}\hbar\omega}_{\text{"quantum"}} + \underbrace{\frac{1}{2}m\omega^2 s_0^2}_{\text{due to the displacement}}.$$

NOTE: $|s_0\rangle$ is <u>not</u> an energy eigenstate!

- Time evolution: $|s_0\rangle \longrightarrow |s_0, t\rangle$

Heisenberg picture

- $\langle s_0, t|\tilde{x}|s_0, t\rangle = \langle s_0|\tilde{x}(t)|s_0\rangle$

$$= \langle s_0|\,\tilde{x}(0)\cos\omega t + \cancel{\frac{\tilde{p}(0)}{m\omega}}^{\,0}\sin\omega t\,|s_0\rangle$$

$$= s_0\cos\omega t.$$

- $\langle s_0, t|\tilde{p}|s_0, t\rangle = \langle s_0|\tilde{p}(t)|s_0\rangle$

$$= -m\omega s_0 \sin\omega t.$$

$\therefore$ <u>$\langle\tilde{x}\rangle_{s_0}$ and $\langle\tilde{p}\rangle_{s_0}$ are oscillating</u> as their classical counterparts do.

- $$\langle(\Delta\tilde{x})^2\rangle_{s_0} = \langle s_0|\tilde{x}^2(t)|s_0\rangle - \langle s_0|\tilde{x}(t)|s_0\rangle^2$$

$$= \langle s_0|\,\tilde{x}(0)^2\cos^2\omega t + \frac{\tilde{p}(0)^2}{(m\omega)^2}\sin^2\omega t + \bigcirc[\tilde{x},\tilde{p}]\,|s_0\rangle$$

$$- s_0^2\cos^2\omega t \qquad \langle[\tilde{x},\tilde{p}]\rangle_{s_0} = 0.$$

$$= \left(\cancel{s_0^2} + \frac{\hbar}{2m\omega}\right)\cos^2\omega t + \frac{m\hbar\omega}{2}\cdot\frac{1}{(m\omega)^2}\sin^2\omega t$$

$$\cancel{- s_0^2\cos^2\omega t}$$

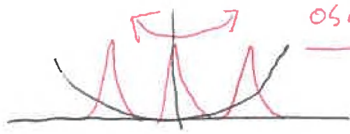
$$= \frac{\hbar}{2m\omega} \quad // . \qquad (t\text{-indep}) \quad : \text{It's the same as in } |0\rangle \text{ !!!}$$

$$\langle(\Delta\tilde{p})^2\rangle_{s_0} = \frac{m\hbar\omega}{2} \quad //$$

$$\Rightarrow \quad \langle (\Delta \tilde{x})^2 \rangle_{s_0} \langle (\Delta \tilde{p})^2 \rangle_{s_0} = \frac{\hbar^2}{4}$$; minimum uncertainty !

(Gaussian wave packet).

So, It looks like



"oscillating"

• $|s_0\rangle$ in the energy eigenkets as base kets.

$$|s_0\rangle = \exp\left(-\frac{i \tilde{p} s_0}{\hbar}\right)|0\rangle = \exp\left[\frac{s_0}{\sqrt{2}x_0}(\underset{A}{\tilde{a}^+} - \underset{B}{\tilde{a}})\right]|0\rangle$$

$$\tilde{p} = i\sqrt{\frac{m\hbar\omega}{2}}(-\tilde{a} + \tilde{a}^+) \qquad \Big\| \quad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

using the case in the Baker-Campbell-Hausdorff theorem.

$$\left[\quad e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \right.$$

$$\left. \text{provided that } [A,[A,B]] = [B,[A,B]] = 0 \quad \right]$$

$$|s_0\rangle = e^{\frac{s_0}{\sqrt{2}x_0}\tilde{a}^+} \underbrace{e^{-\frac{s_0}{\sqrt{2}x_0}\tilde{a}}}_{1+\cancel{O(\tilde{a})}} \underbrace{e^{-\frac{1}{4}\frac{s_0^2}{x_0^2}}}_{c\text{-number}} |0\rangle.$$

$\rightarrow$ 0 with $|0\rangle$.

$$= e^{-\frac{1}{4}\frac{s_0^2}{x_0^2}} \exp\left[\frac{s_0}{\sqrt{2}x_0}\tilde{a}^+\right]|0\rangle$$

$$\Rightarrow |s_0\rangle = e^{-\frac{1}{4}\frac{s_0^2}{x_0^2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{s_0}{\sqrt{2}x_0}\right)^n (\tilde{a}^+)^n |0\rangle$$

$$= e^{-\frac{1}{4}\frac{s_0^2}{x_0^2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\left(\frac{s_0}{\sqrt{2}x_0}\right)^n |n\rangle.$$

$$\equiv \sum_{n=0}^{\infty} c_n |n\rangle. \qquad \Big\| \quad c_n = e^{-\frac{1}{4}\frac{s_0^2}{x_0^2}} \frac{1}{\sqrt{n!}}\left(\frac{s_0}{\sqrt{2}x_0}\right)^n.$$

- prob. for $|s_0\rangle$ to be $|n\rangle$ $= |C_n|^2$

$$= \exp\left[-\frac{1}{2}\frac{s_0^2}{x_0^2}\right]\cdot\frac{1}{n!}\left(\frac{s_0}{\sqrt{2}x_0}\right)^{2n}$$

$$\equiv e^{-\bar{n}}\frac{\bar{n}^n}{n!} \quad \Big\| \quad \bar{n} = \frac{s_0^2}{2x_0^2}$$

$\leftarrow$ This is the Poisson distribution with mean $\bar{n}$.

$\bar{n}$ = expectation value of n $= \sum_{n=0}^{\infty} n|C_n|^2 = \sum_{n=0}^{\infty} n\, e^{-\bar{n}}\frac{\bar{n}^n}{n!}$

$$= e^{-\bar{n}}\,\bar{n}\frac{d}{d\bar{n}}\Big(\underbrace{\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}}_{=e^{\bar{n}}}\Big)$$

$\updownarrow$

- $\langle s_0|\hat{N}|s_0\rangle = \sum_{n,m} C_n^* C_m\, n\,\delta_{n,m} = \bar{n}$.

$\Rightarrow \quad \langle E\rangle = \hbar\omega\left(\bar{n}+\frac{1}{2}\right) \approx \hbar\omega\bar{n} \quad \Big\| \quad s_0^2 \gg x_0^2$

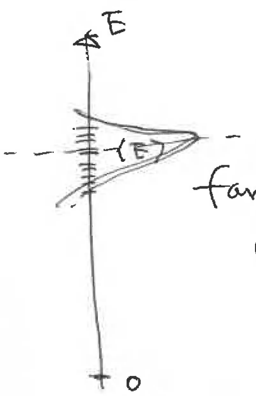
- Energy uncertainty

$$\Delta E = \hbar\omega\left[\overline{n^2}-\bar{n}^2\right]^{\frac{1}{2}} = \hbar\omega\frac{s_0}{2x_0}$$

$$\Rightarrow \quad \frac{\Delta E}{\hbar\omega} = \frac{s_0}{2x_0} \gg 1 \qquad \text{(classical limit)}$$

But, $\dfrac{\langle E\rangle}{\Delta E} = \dfrac{s_0}{2x_0} \gg 1$

: $\langle E\rangle$ is well defined!



far from zero, But narrow enough!

- Generalization

$$|S_0\rangle = \exp\left(-\frac{i\tilde{p}S_0}{\hbar}\right)|0\rangle = \exp\left[\frac{S_0}{\sqrt{2}x_0}(\tilde{a}^+ - \tilde{a})\right]|0\rangle$$

↳ generalization

$$|\alpha\rangle = \underset{\text{unitary}}{D(\alpha)}\,|0\rangle = \exp(\alpha\tilde{a}^+ - \alpha^*\tilde{a})\,|0\rangle.$$

It generates a "displacement" in "generalized" coordinates.

what's $|\alpha\rangle$?

c-number (complex)

$$\tilde{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \quad : \text{eigenket of } \tilde{a}.$$

proof.

$$\tilde{a}\,|\alpha\rangle = \tilde{a}\,\exp(\alpha\tilde{a}^+ - \alpha^*\tilde{a})\,|0\rangle$$

$$= [\underbrace{\tilde{a}}_{=A},\, e^{\overbrace{\alpha\tilde{a}^+ - \alpha^*\tilde{a}}^{=B}}]\,|0\rangle \qquad \| \ \tilde{a}\,|0\rangle = 0.$$

using $[A, e^B] = [A,B]e^B$ when $[[A,B],B] = 0$.

$$= \alpha\,|\alpha\rangle.$$

→ expectation values

$$\langle\alpha|\tilde{x}|\alpha\rangle = \frac{x_0}{\sqrt{2}}\langle\alpha|\tilde{a} + \tilde{a}^+|\alpha\rangle = \frac{x_0}{\sqrt{2}}(\alpha + \alpha^*) = \sqrt{2}x_0 \,\mathrm{Re}[\alpha]$$

$$\langle\alpha|\tilde{p}|\alpha\rangle = \sqrt{2}\hbar x_0^{-1}\,\mathrm{Im}[\alpha]$$

→ Time-evolution

$$|\alpha, t\rangle = \underbrace{e^{-\frac{iHt}{\hbar}}\, e^{\alpha\tilde{a}^+ - \alpha^*\tilde{a}}\, e^{\frac{iHt}{\hbar}}}\, e^{-\frac{iHt}{\hbar}}\,|0\rangle$$

$$= e^{\alpha\tilde{a}^+(-t) - \alpha^*\tilde{a}(-t)}\, e^{-\frac{i\omega t}{2}}\,|0\rangle$$

$$\to (\alpha e^{-i\omega t})\tilde{a}^+ - (\alpha e^{-i\omega t})^*\tilde{a}$$

and

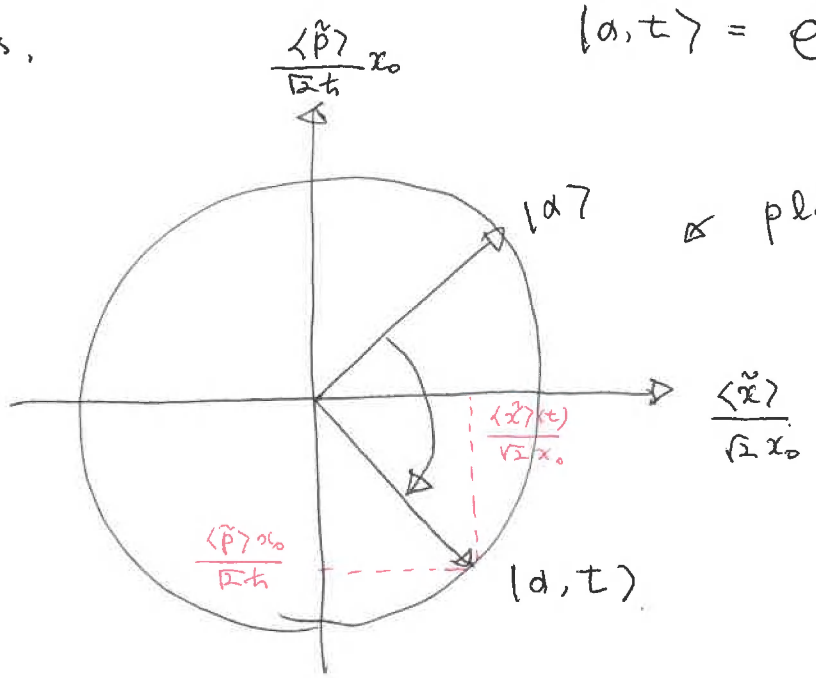
$$\tilde{a}^+(-t) = e^{-i\omega t}\,\tilde{a}^+$$

$$\tilde{a}(-t) = e^{i\omega t}\,\tilde{a}$$

Thus,

$$|\alpha, t\rangle = e^{-i\frac{\omega t}{2}}\,|e^{-i\omega t}\alpha\rangle$$

Thus, $|\alpha, t\rangle = e^{-i\frac{\omega t}{2}} |e^{-i\omega t}\alpha\rangle$



α plane of complex α.

$\longleftrightarrow$ "classical" phase diagram in the generalized coordinates. (p, q).

Supplement

- properties of $D(\alpha)$

① $D^{+}(\alpha) = D^{-1}(\alpha) = D(-\alpha)$: unitarity

② $D^{+}(\alpha)\, \tilde{a}\, D(\alpha) = \tilde{a} + \alpha$

③ $D^{+}(\alpha)\, \tilde{a}^{+}\, D(\alpha) = \tilde{a}^{+} + \alpha^{*}$

④ $D(\alpha+\beta) = D(\alpha) D(\beta) \cdot \exp[-i\, \mathrm{Im}(\alpha\beta^{*})]$

- $\{|\alpha\rangle\}$ is non-orthogonal basis.

$$|\langle\alpha|\beta\rangle| = \exp[-|\alpha-\beta|]^{2} \neq 0 \quad \text{for } \alpha \neq \beta.$$

- Still, $\{|\alpha\rangle\}$ has a completeness relation (overcomplete!)

$$\int \frac{d^{2}\alpha}{\pi} |\alpha\rangle\langle\alpha| = \mathbb{1}$$